Minnesota State High School Mathematics League

2014-15 State Tournament, Invitational Event

SOLUTIONS (page 1 of 3)

Quickies (8 points)

$\boxed{2\sqrt{2}}$ *or* $\boxed{2^{\frac{3}{2}}}$

1. Determine exactly the value of $\sqrt[7]{64}\cdot\sqrt[14]{64}\cdot\sqrt[28]{64}$.

$\sqrt[7]{2^6}\cdot\sqrt[14]{2^6}\cdot\sqrt[28]{2^6}=2^{\frac{6}{7}}\cdot2^{\frac{6}{14}}\cdot2^{\frac{6}{28}}=2^{\frac{42}{28}}=2^{\frac{3}{2}}.$

$Area=\boxed{84}$

2. Alec glued together two right triangles with integer side lengths to form a third triangle, one whose sides measured 10, 17, and 21. Calculate the area of this triangle.

Treating 21 as the base, we have a 6-8-10 glued to an 8-15-17, with shared height 8. Area = $\frac{1}{2}\cdot8\cdot21=84$.

$x=\boxed{\frac{4}{19}}$

3. Determine exactly the value of x if $\frac{4x-1}{2x+1}=\frac{2x-1}{x+5}$.

Cross multiplying yields $(4x-1)(x+5)=(2x+1)(2x-1)\Rightarrow 4x^2+19x-5=4x^2-1\Rightarrow x=\frac{4}{19}$.

$\sqrt{x-y}=\boxed{21}$

4. If $\sqrt{\sqrt{x}+\sqrt{y}}=7$ and $\sqrt{\sqrt{x}-\sqrt{y}}=3$, what is the value of $\sqrt{x-y}$?

$\sqrt{\sqrt{x}+\sqrt{y}}\cdot\sqrt{\sqrt{x}-\sqrt{y}}=\sqrt{\left(\sqrt{x}\right)^2-\left(\sqrt{y}\right)^2}=\sqrt{x-y}=7\cdot3=21.$

$\boxed{1}$

5. The values of a, b, and c are such that the polynomial $p(x)=(ax^2+bx+c)(cx^2+bx+a)$ has 4 real positive roots. Determine exactly the <u>product</u> of these roots.

The product of the roots will be the constant term divided by the x^4 *coefficient:* $\frac{ac}{ca}=1$.

$\boxed{\frac{\sqrt{3}}{2}}$

6. The line $\sqrt{3}\cdot x-2y=6$ forms an acute angle with the positive x-axis. If the measure of this angle is $\tan^{-1}n$, determine n exactly.

The slope of the given line is $\frac{\sqrt{3}}{2}$*, which is equal to the desired tangent value.*

$\boxed{14}$

7. The roots of a quadratic equation are $\frac{1}{3}$ and $-\frac{2}{5}$. If the quadratic's coefficients are relatively prime integers and the coefficient of the x^2 term is positive, what is the sum of the coefficients?

$(3x-1)(5x+2)=15x^2+1x-2$*, whose coefficients sum to* 14.

$\sin\frac{\theta}{2}=\boxed{\frac{\sqrt{30}}{6}}$

8. If $\frac{\pi}{2}<\theta<\pi$ and $\sin\theta=\frac{\sqrt{5}}{3}$, determine $\sin\frac{\theta}{2}$ exactly.

$\cos\theta=\pm\sqrt{1-\sin^2\theta}=\pm\sqrt{1-\frac{5}{9}}=\pm\frac{2}{3}.\quad \frac{\pi}{2}<\theta<\pi\Rightarrow\cos\theta=-\frac{2}{3}$, *so* $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{5}{6}}=\frac{\sqrt{30}}{6}$.

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Problems (8 points)

Area = $30\sqrt{2}$

9. The lengths of the tangents from the vertices of a triangle to its inscribed circle are 4, 5, and 6. Determine exactly the area of this triangle.

$c = \frac{13}{7}$

10. Determine exactly the <u>greatest</u> possible value of c if $a^2+b^2+c^2=6, a+b+c=4, and\ a=2b.$

$m\angle A = 196°$

11. Find the <u>least</u> positive angle A (in degrees) for which $\tan 37° = \sqrt{\frac{1+\sin A}{1-\sin A}}$.

$[-2,1] \cup [1,3)$

Graders: Deduct 1 point for each missing or incorrect interval; min. score is 0.

12. Let $\lfloor x \rfloor$ = the greatest integer less than or equal to *x*.

Describe the set of <u>all</u> real numbers *x* such that $1 \le x\lfloor x \rfloor \le 6$.

Challenges (8 points)

$\frac{81}{25}$

13. *(2 pts)* Determine the exact value of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (9)^{n+1}}{4^{2n}}$.

$\frac{b}{a} = 1, \frac{2}{3}, \frac{3}{2}$

Graders: Deduct 1 point for each missing or incorrect value; min. score is 0.

14. *(3 pts)* Let *a* and *b* be real numbers. Determine exactly all values of the ratio $\frac{b}{a}$ such that $\frac{ax+3}{bx+2} = \frac{ax-2}{bx-3}$ has no real solutions.

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15. *(3 pts)* In $\triangle ABC$, altitude $\overline{AD}$ and sides $\overline{AB}$ and $\overline{BC}$ all have integer lengths. If $AC = 53$, find the <u>smallest</u> possible perimeter for $\triangle ABC$.

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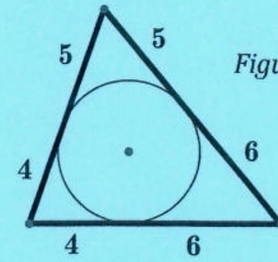

Figure 9

9. *See Figure 9. The side lengths of the circumscribed triangle are 9, 10, and 11. Using Heron's formula,* $\sqrt{15(15-9)(15-10)(15-11)} = \sqrt{15\cdot 6\cdot 5\cdot 4} = \sqrt{4\cdot 3^2\cdot 5^2\cdot 2} = \boxed{30\sqrt{2}}$.

10. $c = 4-a-b = 4-(2b)-b = 4-3b$. *By substitution,* $(2b)^2+b^2+(4-3b)^2 = 6 \Rightarrow 7b^2-12b+5=0 \Rightarrow (7b-5)(b-1)=0$ $\Rightarrow b = \frac{5}{7}$ *or* $b=1$. *The lesser of these b values will generate the greater c value:* $c = 4-3\left(\frac{5}{7}\right) = \boxed{\frac{13}{7}}$.

11. $\tan 37° = \tan\left(\frac{74°}{2}\right) = \sqrt{\frac{1-\cos 74°}{1+\cos 74°}}$. *However, we would rather see sine values underneath that radical. By cofunction identity,* $\cos 74° = \sin 16° = -\sin(-16°)$, *and the least positive angle with that sine value is* $180°+16° = \boxed{196°}$.

12. *For x-values larger than 0, x cannot be less than 1, because then* $\lfloor x \rfloor$ *would be 0. We also may not use* $x \ge 3$, *because then* $x\lfloor x \rfloor$ *would yield values of at least 9. For x-values less than 0, x cannot be less than –2, because the product* $x\lfloor x \rfloor$ *would yield values larger than 6. Furthermore,* $-1 < x < 0$ *yields products less than 1. Therefore,* $\boxed{-2 \le x \le -1 \text{ or } 1 \le x < 3}$.

13. *Rewrite* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot 9^{(n+1)}}{4^{2n}}$ *as* $-9\sum_{n=1}^{\infty}\left(-\frac{9}{16}\right)^n$. *This is a geometric series whose first term is* $\frac{81}{16}$ *and whose common ratio is* $-\frac{9}{16}$. *The infinite sum of this series is then* $\frac{\frac{81}{16}}{1-\left(-\frac{9}{16}\right)} = \frac{81}{16}\cdot\frac{16}{25} = \boxed{\frac{81}{25}}$.

14. *First note that* $x \ne -\frac{2}{b}$ *or* $\frac{3}{b}$. $\frac{ax+3}{bx+2} = \frac{ax-2}{bx-3} \Rightarrow (ax+3)(bx-3) = (bx+2)(ax-2)$, *which becomes* $abx^2-3ax+3bx-9 = abx^2-2bx+2ax-4 \Rightarrow 5bx-5ax=5 \Rightarrow (b-a)x=1$. *Thus, either* $b \ne a$ $\left(\frac{b}{a}=1\right)$, *or* $x = \frac{1}{b-a}$. *Because* $x \ne -\frac{2}{b}$, *we set* $\frac{1}{b-a} = -\frac{2}{b}$ *and obtain* $b = 2a-2b \Rightarrow 3b = 2a \Rightarrow \frac{b}{a} = \frac{2}{3}$. *Similarly, because* $x \ne \frac{3}{b}$, *we set* $\frac{1}{b-a} = \frac{3}{b} \Rightarrow b = 3a-2b \Rightarrow \frac{b}{a} = \frac{3}{2}$. *Therefore, the ratios of* $\frac{b}{a}$ *that give no solutions are* $\boxed{1, \frac{2}{3}, \text{and } \frac{3}{2}}$.

15. *Note that* $\triangle ABC$ *may be constructed either by building Pythagorean triangles on either side of* $\overline{AD}$ *(making* $\triangle ABC$ *acute), or by "nesting" one Pythagorean triangle inside the other, with both sharing "right leg"* $\overline{AD}$ *(making* $\triangle ABC$ *obtuse, and* $\overline{AD}$ *an exterior altitude). Since we're trying to minimize the perimeter, we probably will want to consider the "nesting" case, where 53 is the hypotenuse of both nested Pythagorean triangles. The only Pythagorean triple containing 53 as the hypotenuse is 28/45/53, so AD = 28 or 45. For either choice, look for Pythagorean triples that share that leg, and for which the other leg makes the perimeter as small as possible. If AD = 45, this triple is 24/45/51, which gives a triangle of side lengths 53, 51, and 4. If AD = 28, this triple is 21/28/35, which gives a triangle of side lengths 53, 35, 24. The smaller perimeter occurs in the first case, where P =* $\boxed{108}$.